Lecture 18: Maximum and Minimum Values 4.1

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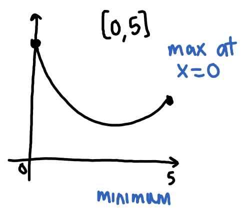
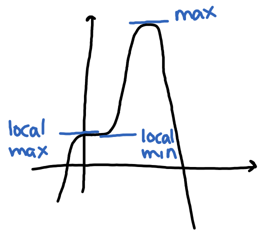




absolute max/min: at a point $x = c$: $f(c) \geq f(x)$ for all x in the doman
$\leq$

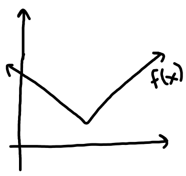
local max/min: at a point $x = c$: $f(x) \geq f(x)$ for x near c
$\leq$

Extreme value theorem

If f is continuous on a closed interval [a,b], then f attains an absolute minimum and maximum at points c and d in [c,d].
→ edge points are taken into account on a closed interval

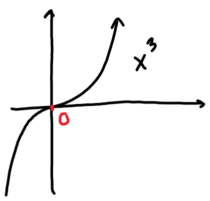
Fermat's theorem

If f has a local min or max at c and if $f'(c)$ exists, then $f'(c) = 0$



local min but f'(c) DNE

Ex Find $f'(0)$
$f(x) = x^3, f'(x) = 3x^2$
$\rightarrow 3x^2 = 0$ if $x = 0$



f(x) is strictly increasing, so f(0) isn't a min or max

Just because $f'(c) = 0$, it doesn't necessarily mean c is a min or max.

A **critical point** of a function f is a point c such that $f'(c) = 0$ OR $f'(x)$ DNE.

Example: Find the critical points of $f(x)$

$$f(x) = x^{\frac{3}{5}} \cdot (4 - x)$$

$$f'(x) = \frac{3}{5} x^{\frac{-2}{5}} \cdot (4 - x) + x^{\frac{3}{5}}(-1)$$

find critical points → set $f'(x) = 0$, solve for x

$$f'(x) = \frac{3}{5} x^{-\frac{2}{5}} \cdot (4 - x) - x^{\frac{3}{5}} = 0$$ factor out $x^{3/5}$

$$= x^{\frac{3}{5}} \left(\frac{3}{5} x^{-1} \cdot (4 - x) - 1 \right) = 0$$

$$\frac{3}{5x}(4 - x) - 1 = 0 \quad \Rightarrow \quad 4 = \frac{8x}{3} \quad \Rightarrow \quad x = \frac{3}{2} \quad \text{and} \quad x = 0.$$

At $x = 0$: $x = 0$ is not in the doman of $f'(x)$, so $f'(0)$ DNE but $x = 0$ is still a critical point.

Answer: critical points of $f(x)$ are at $x = 0,\ x = \frac{3}{2}$.

How to find absolute min and max of f on [a,b]

i) Find critical points x_i of f on $[a, b]$
ii) Evaluate $f(a), f(x_i)$ for all $i, f(b)$
iii) Compare all these values

Example Find absolute min/max of $f(x) = -x^4 + 4x^3 - 2x^2 + 7$ on $[0,4]$.

1) Find critical points.

$f'(x) = -4x^3 + 12x^2 - 4x$

$0 = -4x(x^2 - 3x + 1)$

$0 = x^2 - 3x + 1$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x_{1,2} = \frac{3 \pm \sqrt{5}}{2}, x_3 = 0$$

$\Rightarrow$ critical points: $0, \frac{3+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}$

2) Compute the values.

$f(0) = 7, f(4) = -25$ endpoints

$f\left(\frac{3+\sqrt{5}}{2}\right) = 18.090, f\left(\frac{3-\sqrt{5}}{2}\right) = 6.190$

3) Choose the min and max

We have an absolute min at $x = 4$, the min is $f(4) = -25$

We have an absolute max at $x = \frac{3+\sqrt{5}}{2}$, the max is $f\left(\frac{3+\sqrt{5}}{2}\right) = 18.09$

Edge points are never local min or max

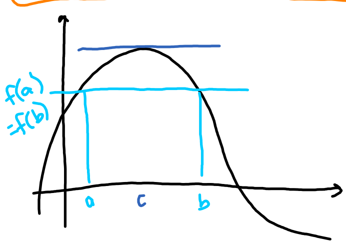
Mean Value Theorem 4.2

Rolle's Theorem

Let f be

1) continuous on $[a, b]$
2) differentiable on (a, b)
3) $f(a) = f(b)$

Then, there exists $c \in$ (a, b) such that $f'(c) = 0$.



if the edge points are the same, then the slope changes the sign (or stays constant (always 0 - constant function))

Idea: extreme value theorem gives you the min and max. if:

1) $f(x) > f(a)$: you have a local max
2) $f(x) < f(a)$: you have a local min

$\Rightarrow$ c is a local max or min, then $f'(c) = 0$ by Fermat's theorem.